Multiple Soliton Solutions for a Variety of Coupled Modified Korteweg–de Vries Equations

Abdul-Majid Wazwaz

Department of Mathematics, Saint Xavier University Chicago, IL 60655

Reprint requests to A.-M. W.; E-mail: wazwaz@sxu.edu

Z. Naturforsch. **66a,** 625 – 631 (2011) / DOI: 10.5560/ZNA.2011-0034
Received May 19, 2011 / revised July 6, 2011

We make use of Hirota's bilinear method with computer symbolic computation to study a variety of coupled modified Korteweg–de Vries (mKdV) equations. Multiple soliton solutions and multiple singular soliton solutions are obtained for each coupled equation. The resonance phenomenon of each coupled mKdV equation is proved not to exist.

Key words: Coupled mKdV Equation; Hirota Bilinear Method; Hietarinta Approach; Multiple Soliton Solutions; Multiple Singular Soliton Solutions; Resonance.

1. Introduction

Recently, many nonlinear coupled evolution equations, such as the coupled Korteweg–de Vries (KdV) equation, the coupled Boussinesq equation, and the coupled mKdV equation, appear in scientific applications [1 – 13]. The coupled evolution equations attracted a considerable research work in the literature. The aims of these works have been the determination of soliton solutions and the proof of complete integrability of these coupled equations [14 – 24].

Various methods have been used to investigate the nonlinear evolution and the coupled nonlinear evolution equations. Examples of the methods that have been used are the Hirota bilinear method, the Hietarinta approach, the Bäcklund transformation method, the Darboux transformation, the Pfaffian technique, the inverse scattering method, the Painlevé analysis, the generalized symmetry method, and other methods. The Hirota bilinear method [1 – 7], the Hietarinta approach [8, 9], and the Hereman simplified form [10 – 12] are rather heuristic and significant. These approaches possess powerful features that make them practical for the determination of multiple soliton solutions [13 – 24] for a wide class of nonlinear evolution equations. The computer symbolic systems such as Maple and Mathematica allow us to perform complicated and tedious calculations.

It is interesting to point out that the soliton solution should demonstrate a wave of permanent form. The soliton solution is localized, which means that the solution either decays exponentially to zero such as the KdV solitons, or converges to a constant at infinity such as the sine-Gordon equation. Since we will talk about the multiple soliton solutions, we point out that the soliton interacts with other solitons preserving its character. We also add that the soliton solution $u(x,t)$, along with its derivatives, tends to zero as $|x| \to \infty$. This clearly shows that the soliton reside in Hilbert space, and it results from initial-boundary value problems.

Concerning the modified KdV equation, it describes nonlinear wave propagation in systems with polarity symmetry. The mKdV equation is used in electrodynamics, wave propagation in size quantized films, and in elastic media. It is used to describe acoustic waves in anharmonic lattices and Alfvén waves in collisionless plasma. The mKdV equation differs from the KdV equation only because of its cubic nonlinearity. The mKdV equation is completely integrable and can be solved by the inverse scattering method.

In this work, a variety of coupled mKdV equations will be investigated for complete integrability and for the determination of multiple soliton solutions. The coupled mKdV equations that we selected are

$$\begin{aligned} &u_t + 6\alpha uvu_x + u_{xxx} = 0, \\ &v_t + 6\alpha uvv_x + v_{xxx} = 0, \end{aligned} \tag{1}$$

$$\begin{aligned} &u_t + 6\alpha uvv_x + 6u^2u_x - 6v^2u_x + u_{xxx} = 0, \\ &v_t + 24\alpha uvu_x + 6u^2v_x - 6v^2v_x + v_{xxx} = 0, \end{aligned} \tag{2}$$

and

$$u_t+\alpha(v^2u_x-u^2u_x)+\frac{\alpha}{4}u_{xxx}=0,\\ v_t+\alpha(v^2v_x-u^2v_x)+\frac{\alpha}{4}v_{xxx}=0. \tag{3}$$

The Hirota bilinear method [1–7], the Hietarinta approach [8, 9], and Hereman's simplified form [10–12] will be used in this work. Our goal is to construct multiple regular soliton solutions and multiple singular soliton solutions for each coupled equation. Hirota and Ito in [2] examined the phenomena of two solitons near the resonant state, two solitons at the resonant state, and two solitons after colliding with each other. The systems of mKdV equations (1)–(3) will be tested for resonance effects.

2. The First Coupled mKdV Equation

We first study the coupled mKdV equation given by

$$u_t+6\alpha uvu_x+u_{xxx}=0,\\ v_t+6\alpha uvv_x+v_{xxx}=0, \tag{4}$$

where α is a constant. For $u=v$ the system (1) becomes the mKdV equation. This system was studied first by Hirota [1], then by others.

2.1. Multiple Soliton Solutions

Substituting

$$u(x,t)=\mathrm{e}^{\theta_i},\ \ \theta_i=k_ix-c_it,\\ v(x,t)=A\mathrm{e}^{\theta_i}, \tag{5}$$

where A is a constant, into the linear terms of (4) gives the dispersion relation by

$$c_i=k_i^3, \tag{6}$$

and as a result we obtain

$$\theta_i=k_ix-k_i^3t. \tag{7}$$

The multi-soliton solutions of the coupled mKdV equation are given by

$$u(x,t)=R\left(\arctan\left(\frac{f(x,t)}{g(x,t)}\right)\right)_x=R\frac{f_xg-g_xf}{f^2+g^2},\\ v(x,t)=R_1\left(\arctan\left(\frac{f(x,t)}{g(x,t)}\right)\right)_x=R_1\frac{f_xg-g_xf}{f^2+g^2}, \tag{8}$$

where the auxiliary functions $f(x,t)$ and $g(x,t)$ for the single soliton solution are given by

$$f(x,t)=\mathrm{e}^{\theta_1}=\mathrm{e}^{k_1x-k_1^3t},\\ g(x,t)=1. \tag{9}$$

Substituting (8) and (9) into (4) and solving for R and R_1, we find

$$R=\beta,\\ R_1=\frac{4}{\alpha\beta^2}, \tag{10}$$

where β is a constant.

Combining (8)–(10) gives the single soliton solution

$$u(x,t)=\frac{\beta k_1\,\mathrm{e}^{k_1\left(x-k_1^2t\right)}}{1+\mathrm{e}^{2k_1\left(x-k_1^2t\right)}},\\ v(x,t)=\frac{4k_1\,\mathrm{e}^{k_1\left(x-k_1^2t\right)}}{\alpha\beta\left(1+\mathrm{e}^{2k_1\left(x-k_1^2t\right)}\right)}. \tag{11}$$

The last result determines the relation between $u(x,t)$ and $v(x,t)$ by

$$\frac{u(x,t)}{v(x,t)}=\frac{\alpha\beta^2}{4}. \tag{12}$$

To determine the two-soliton solutions, we set

$$f(x,t)=\mathrm{e}^{\theta_1}+\mathrm{e}^{\theta_2}=\mathrm{e}^{k_1\left(x-k_1^2t\right)}+\mathrm{e}^{k_2\left(x-k_2^2t\right)},\\ g(x,t)=1-a_{12}\mathrm{e}^{\theta_1+\theta_2}=1-a_{12}\mathrm{e}^{(k_1+k_2)x-\left(k_1^3+k_2^3\right)t}. \tag{13}$$

Substituting (13) into (8) and using the obtained result in the coupled mKdV equation (4), one obtains the phase shift a_{12} by

$$a_{12}=\frac{(k_1-k_2)^2}{(k_1+k_2)^2}, \tag{14}$$

and this can be generalized to

$$a_{ij}=\frac{(k_i-k_j)^2}{(k_i+k_j)^2},\ \ 1\le i<j\le 3. \tag{15}$$

The two-soliton solutions are obtained by substituting (14) and (13) into (8). It is interesting to point out that (1) does not show any resonant phenomenon [2] because the phase shift term a_{12} in (14) cannot be 0 or ∞ for $|k_1| \neq |k_2|$.

To determine the three-soliton solutions, we set

$$\begin{aligned} f(x,t) &= e^{\theta_1} + e^{\theta_2} + e^{\theta_3} - a_{12}a_{13}a_{23}e^{\theta_1+\theta_2+\theta_3} \\ &= e^{k_1\left(x-k_1^2 t\right)} + e^{k_2\left(x-k_2^2 t\right)} + e^{k_3\left(x-k_3^2 t\right)} \\ &\quad - a_{12}a_{13}a_{23}e^{(k_1+k_2+k_3)x-\left(k_1^3+k_2^3+k_3^3\right)t}, \\ g(x,t) &= 1 - a_{12}e^{\theta_1+\theta_2} - a_{13}e^{\theta_1+\theta_3} - a_{23}e^{\theta_2+\theta_3} \\ &= 1 - a_{12}e^{(k_1+k_2)x-\left(k_1^3+k_2^3\right)t} \\ &\quad - a_{13}e^{(k_1+k_3)x-\left(k_1^3+k_3^3\right)t} \\ &\quad - a_{23}e^{(k_2+k_3)x-\left(k_2^3+k_3^3\right)t}, \end{aligned} \tag{16}$$

where the phase shifts a_{ij} are derived above in (15). The three-soliton solutions for the coupled mKdV equation (4) are obtained by substituting (16) into (8). It is obvious that N-soliton solutions can be obtained for finite N, where $N \geq 1$.

2.2. Singular Soliton Solutions

In this section, we will determine multiple singular soliton solutions for the coupled mKdV equation (4). Following [13], the singular soliton solution of the coupled mKdV equation (4) is assumed to be of the form

$$\begin{aligned} u(x,t) &= R\left(\ln\left(\frac{f(x,t)}{g(x,t)}\right)\right)_x = R\frac{gf_x - fg_x}{gf}, \\ v(x,t) &= R_1\left(\ln\left(\frac{f(x,t)}{g(x,t)}\right)\right)_x = R_1\frac{gf_x - fg_x}{gf}, \end{aligned} \tag{17}$$

where R and R_1 are constants that will be determined. The auxiliary functions $f(x,t)$ and $g(x,t)$ have expansions of the form

$$\begin{aligned} f(x,t) &= 1 + \sum_{n=1}^{\infty} f_n(x,t), \\ g(x,t) &= 1 - \sum_{n=1}^{\infty} g_n(x,t). \end{aligned} \tag{18}$$

Following the discussion presented in the previous section, the dispersion relation is given by

$$c_i = k_i^3, \tag{19}$$

and as a result we obtain

$$\theta_i = k_i x - k_i^3 t. \tag{20}$$

The obtained results give a new definition to (18) in the form

$$\begin{aligned} f(x,t) &= 1 + e^{k_1\left(x-k_1^2 t\right)}, \\ g(x,t) &= 1 - e^{k_1\left(x-k_1^2 t\right)}. \end{aligned} \tag{21}$$

Substituting (21) into (17), and using the outcome in (4), one obtains

$$\begin{aligned} R &= \beta, \\ R_1 &= -\frac{1}{\alpha\beta^2}, \end{aligned} \tag{22}$$

where β is a constant. Combining the previous results, the singular soliton solutions

$$\begin{aligned} u(x,t) &= \frac{2\beta k_1 e^{k_1\left(x-k_1^2 t\right)}}{1 - e^{k_1\left(x-k_1^2 t\right)}}, \\ v(x,t) &= -\frac{2k_1 e^{k_1\left(x-k_1^2 t\right)}}{\alpha\beta\left(1 - e^{k_1\left(x-k_1^2 t\right)}\right)} \end{aligned} \tag{23}$$

are readily obtained. It is clear that

$$\frac{u(x,t)}{v(x,t)} = -\alpha\beta^2. \tag{24}$$

To determine the singular two-soliton solutions, we set

$$\begin{aligned} f(x,t) &= 1 + e^{\theta_1} + e^{\theta_2} + a_{12}e^{\theta_1+\theta_2}, \\ g(x,t) &= 1 - e^{\theta_1} - e^{\theta_2} + b_{12}e^{\theta_1+\theta_2}. \end{aligned} \tag{25}$$

Substituting (25) into (18) and using the outcome into (4), we find that (25) is a solution of this equation if the phase shifts a_{12} and b_{12}, and therefore a_{ij} and b_{ij}, are equal and given by

$$a_{ij} = b_{ij} = \frac{(k_i - k_j)^2}{(k_i + k_j)^2}. \tag{26}$$

For the two-soliton solutions we use $1 \leq i < j \leq 2$ to obtain

$$\begin{aligned} f(x,t) &= 1 + e^{k_1\left(x-k_1^2 t\right)} + e^{k_2\left(x-k_2^2 t\right)} \\ &\quad + \frac{(k_1-k_2)^2}{(k_1+k_2)^2}e^{(k_1+k_2)x-\left(k_1^3+k_2^3\right)t}, \\ g(x,t) &= 1 - e^{k_1\left(x-k_1^2 t\right)} - e^{k_2\left(x-k_2^2 t\right)} \\ &\quad + \frac{(k_1-k_2)^2}{(k_1+k_2)^2}e^{(k_1+k_2)x-\left(k_1^3+k_2^3\right)t}. \end{aligned} \tag{27}$$

This in turn gives the singular two-soliton solutions if we substitute (27) into (17).

To determine the singular three-soliton solutions, we can proceed in a similar manner and set

$$\begin{aligned} f(x,t) &= 1+\mathrm{e}^{\theta_1}+\mathrm{e}^{\theta_2}+\mathrm{e}^{\theta_3}+a_{12}\mathrm{e}^{\theta_1+\theta_2} \\ &\quad +a_{23}\mathrm{e}^{\theta_2+\theta_3}+a_{13}\mathrm{e}^{\theta_1+\theta_3}+f_3(x,t), \\ g(x,t) &= 1-\mathrm{e}^{\theta_1}-\mathrm{e}^{\theta_2}-\mathrm{e}^{\theta_3}+a_{12}\mathrm{e}^{\theta_1+\theta_2} \\ &\quad +a_{23}\mathrm{e}^{\theta_2+\theta_3}+a_{13}\mathrm{e}^{\theta_1+\theta_3}+g_3(x,t). \end{aligned} \tag{28}$$

Substituting (28) into (17) and using the result into (4) to find that

$$\begin{aligned} f_3(x,t) &= b_{123}\mathrm{e}^{\theta_1+\theta_2+\theta_3}, \\ g_3(x,t) &= -b_{123}\mathrm{e}^{\theta_1+\theta_2+\theta_3}, \\ b_{123} &= a_{12}a_{13}a_{23}. \end{aligned} \tag{29}$$

For the singular three-soliton solutions we use $1 \le i < j \le 3$, we therefore obtain

$$\begin{aligned} f(x,t) &= 1+\mathrm{e}^{k_1\left(x-k_1^2t\right)}+\mathrm{e}^{k_2\left(x-k_2^2t\right)}+\mathrm{e}^{k_3\left(x-k_3^2t\right)} \\ &\quad +\frac{(k_1-k_2)^2}{(k_1+k_2)^2}\mathrm{e}^{(k_1+k_2)x-\left(k_1^3+k_2^3\right)t} \\ &\quad +\frac{(k_1-k_3)^2}{(k_1+k_3)^2}\mathrm{e}^{(k_1+k_3)x-\left(k_1^3+k_3^3\right)t} \\ &\quad +\frac{(k_2-k_3)^2}{(k_2+k_3)^2}\mathrm{e}^{(k_2+k_3)x-\left(k_2^3+k_3^3\right)t} \\ &\quad +\frac{(k_1-k_2)^2(k_1-k_3)^2(k_2-k_3)^2}{(k_1+k_2)^2(k_1+k_3)^2(k_2+k_3)^2} \\ &\quad \cdot \mathrm{e}^{(k_1+k_2+k_3)x-\left(k_1^3+k_2^3+k_3^3\right)t}, \\ g(x,t) &= 1-\mathrm{e}^{k_1\left(x-k_1^2t\right)}-\mathrm{e}^{k_2\left(x-k_2^2t\right)}-\mathrm{e}^{k_3\left(x-k_3^2t\right)} \\ &\quad +\frac{(k_1-k_2)^2}{(k_1+k_2)^2}\mathrm{e}^{(k_1+k_2)x-\left(k_1^3+k_2^3\right)t} \\ &\quad +\frac{(k_1-k_3)^2}{(k_1+k_3)^2}\mathrm{e}^{(k_1+k_3)x-\left(k_1^3+k_3^3\right)t} \\ &\quad +\frac{(k_2-k_3)^2}{(k_2+k_3)^2}\mathrm{e}^{(k_2+k_3)x-\left(k_2^3+k_3^3\right)t} \\ &\quad -\frac{(k_1-k_2)^2(k_1-k_3)^2(k_2-k_3)^2}{(k_1+k_2)^2(k_1+k_3)^2(k_2+k_3)^2} \\ &\quad \cdot \mathrm{e}^{(k_1+k_2+k_3)x-\left(k_1^3+k_2^3+k_3^3\right)t}. \end{aligned} \tag{30}$$

The singular three-soliton solutions follow immediately upon substituting (30) into (17).

3. The Second Coupled mKdV Equation

We next study the coupled mKdV equation given by

$$\begin{aligned} &u_t+6\alpha uvv_x+6u^2u_x-6v^2u_x+u_{xxx}=0, \\ &v_t+24\alpha uvu_x+6u^2v_x-6v^2v_x+v_{xxx}=0, \end{aligned} \tag{31}$$

where α is a constant. The discussion here will be parallel to our discussion above.

3.1. Multiple Soliton Solutions

Proceeding as before, the dispersion relation is given by

$$c_i = k_i^3, \tag{32}$$

and we also obtain

$$\theta_i = k_i x - k_i^3 t. \tag{33}$$

The multi-soliton solutions of the coupled mKdV equation (31) are assumed above in (8) where the auxiliary functions $f(x,t)$ and $g(x,t)$ are as given above in (9). Substituting these results into (31) and solving for R and R_1, we find

$$\begin{aligned} R &= \pm\frac{2}{\sqrt{4\alpha-3}}, \\ R_1 &= \pm 2. \end{aligned} \tag{34}$$

Combining these results gives the single soliton solution

$$\begin{aligned} u(x,t) &= \pm\frac{2k_1\,\mathrm{e}^{k_1\left(x-k_1^2t\right)}}{\sqrt{4\alpha-3}\left(1+\mathrm{e}^{2k_1\left(x-k_1^2t\right)}\right)}, \quad \alpha>\frac{3}{4}, \\ v(x,t) &= \pm\frac{4k_1\,\mathrm{e}^{k_1\left(x-k_1^2t\right)}}{\sqrt{4\alpha-3}\left(1+\mathrm{e}^{2k_1\left(x-k_1^2t\right)}\right)}. \end{aligned} \tag{35}$$

The last result determines the relation between $u(x,t)$ and $v(x,t)$ by

$$\frac{u(x,t)}{v(x,t)} = \frac{1}{2}. \tag{36}$$

To determine the two-soliton solutions, we set

$$\begin{aligned} f(x,t) &= \mathrm{e}^{\theta_1} + \mathrm{e}^{\theta_2} = \mathrm{e}^{k_1\left(x-k_1^2 t\right)} + \mathrm{e}^{k_2\left(x-k_2^2 t\right)}, \\ g(x,t) &= 1 - a_{12}\mathrm{e}^{\theta_1+\theta_2} = 1 - a_{12}\mathrm{e}^{(k_1+k_2)x-\left(k_1^3+k_2^3\right)t}. \end{aligned} \tag{37}$$

Substituting (37) into (8) and using the obtained result in the coupled mKdV equation (31), one obtains the phase shift a_{12} by

$$a_{12} = \frac{(k_1-k_2)^2}{(k_1+k_2)^2}, \tag{38}$$

and this can be generalized to

$$a_{ij} = \frac{(k_i-k_j)^2}{(k_i+k_j)^2}, \quad 1 \le i < j \le 3. \tag{39}$$

The two-soliton solutions are obtained by substituting (38) and (37) into (8). It is interesting to point out that (38) does not show any resonant phenomenon [2] because the phase shift term a_{12} in (38) cannot be 0 or ∞ for $|k_1| \neq |k_2|$.

To determine the three-soliton solutions, we use the assumption set in (16). The three-soliton solutions for the coupled mKdV equation (31) are obtained in a like manner to the analysis presented earlier. It is obvious that N-soliton solutions can be obtained for finite N, where $N \ge 1$.

3.2. Singular Soliton Solutions

In this section, we will determine multiple singular soliton solutions for the coupled mKdV equation (31). Following [13], and proceeding as in the previous section, we obtain the following results:

i) The auxiliary functions become

$$\begin{aligned} f(x,t) &= 1 + \mathrm{e}^{k_1\left(x-k_1^2 t\right)}, \\ g(x,t) &= 1 - \mathrm{e}^{k_1\left(x-k_1^2 t\right)}. \end{aligned} \tag{40}$$

Proceeding as before, one obtains

$$\begin{aligned} R &= \frac{1}{\sqrt{5-4\alpha}}, \quad \alpha < \frac{5}{4}, \\ R_1 &= \pm 2. \end{aligned} \tag{41}$$

Combining the previous results, the singular soliton solutions

$$\begin{aligned} u(x,t) &= \pm \frac{2k_1\mathrm{e}^{k_1\left(x-k_1^2 t\right)}}{1-\mathrm{e}^{k_1\left(x-k_1^2 t\right)}}, \\ v(x,t) &= \pm \frac{4k_1\mathrm{e}^{k_1\left(x-k_1^2 t\right)}}{\sqrt{5-4\alpha}\left(1-\mathrm{e}^{k_1\left(x-k_1^2 t\right)}\right)} \end{aligned} \tag{42}$$

are readily obtained. It is clear that

$$\frac{u(x,t)}{v(x,t)} = \pm\frac{1}{2}. \tag{43}$$

ii) To determine the singular two-soliton solutions, we proceed as before to find that the phase shifts a_{12} and b_{12}, and therefore a_{ij} and b_{ij}, are equal and given by

$$a_{ij} = b_{ij} = \frac{(k_i-k_j)^2}{(k_i+k_j)^2}. \tag{44}$$

For the two-soliton solutions, we use $1 \le i < j \le 2$ to obtain

$$\begin{aligned} f(x,t) &= 1 + \mathrm{e}^{k_1\left(x-k_1^2 t\right)} + \mathrm{e}^{k_2\left(x-k_2^2 t\right)} \\ &\quad + \frac{(k_1-k_2)^2}{(k_1+k_2)^2}\mathrm{e}^{(k_1+k_2)x-\left(k_1^3+k_2^3\right)t}, \\ g(x,t) &= 1 - \mathrm{e}^{k_1\left(x-k_1^2 t\right)} - \mathrm{e}^{k_2\left(x-k_2^2 t\right)} \\ &\quad + \frac{(k_1-k_2)^2}{(k_1+k_2)^2}\mathrm{e}^{(k_1+k_2)x-\left(k_1^3+k_2^3\right)t}. \end{aligned} \tag{45}$$

This in turn gives the singular two-soliton solutions if we substitute (45) into (17).

To determine the singular three-soliton solutions, we can proceed in a similar manner and set

$$\begin{aligned} f(x,t) &= 1 + \mathrm{e}^{\theta_1} + \mathrm{e}^{\theta_2} + \mathrm{e}^{\theta_3} + a_{12}\mathrm{e}^{\theta_1+\theta_2} \\ &\quad + a_{23}\mathrm{e}^{\theta_2+\theta_3} + a_{13}\mathrm{e}^{\theta_1+\theta_3} + f_3(x,t), \\ g(x,t) &= 1 - \mathrm{e}^{\theta_1} - \mathrm{e}^{\theta_2} - \mathrm{e}^{\theta_3} + a_{12}\mathrm{e}^{\theta_1+\theta_2} \\ &\quad + a_{23}\mathrm{e}^{\theta_2+\theta_3} + a_{13}\mathrm{e}^{\theta_1+\theta_3} + g_3(x,t). \end{aligned} \tag{46}$$

Substituting (46) into (17) and using the result into (31) to find that

$$\begin{aligned} f_3(x,t) &= b_{123}\mathrm{e}^{\theta_1+\theta_2+\theta_3}, \\ g_3(x,t) &= -b_{123}\mathrm{e}^{\theta_1+\theta_2+\theta_3}, \\ b_{123} &= a_{12}a_{13}a_{23}. \end{aligned} \tag{47}$$

For the singular three-soliton solutions, we use $1 \le i < j \le 3$; we therefore obtain

$$
\begin{aligned}
f(x,t) &= 1 + e^{k_1\left(x-k_1^2 t\right)} + e^{k_2\left(x-k_2^2 t\right)} + e^{k_3\left(x-k_3^2 t\right)} \\
&\quad + \frac{(k_1-k_2)^2}{(k_1+k_2)^2} e^{(k_1+k_2)x-\left(k_1^3+k_2^3\right)t} \\
&\quad + \frac{(k_1-k_3)^2}{(k_1+k_3)^2} e^{(k_1+k_3)x-\left(k_1^3+k_3^3\right)t} \\
&\quad + \frac{(k_2-k_3)^2}{(k_2+k_3)^2} e^{(k_2+k_3)x-\left(k_2^3+k_3^3\right)t} \\
&\quad + \frac{(k_1-k_2)^2(k_1-k_3)^2(k_2-k_3)^2}{(k_1+k_2)^2(k_1+k_3)^2(k_2+k_3)^2} \\
&\quad \cdot e^{(k_1+k_2+k_3)x-\left(k_1^3+k_2^3+k_3^3\right)t}, \\
g(x,t) &= 1 - e^{k_1\left(x-k_1^2 t\right)} - e^{k_2\left(x-k_2^2 t\right)} - e^{k_3\left(x-k_3^2 t\right)} \\
&\quad + \frac{(k_1-k_2)^2}{(k_1+k_2)^2} e^{(k_1+k_2)x-\left(k_1^3+k_2^3\right)t} \\
&\quad + \frac{(k_1-k_3)^2}{(k_1+k_3)^2} e^{(k_1+k_3)x-\left(k_1^3+k_3^3\right)t} \\
&\quad + \frac{(k_2-k_3)^2}{(k_2+k_3)^2} e^{(k_2+k_3)x-\left(k_2^3+k_3^3\right)t} \\
&\quad - \frac{(k_1-k_2)^2(k_1-k_3)^2(k_2-k_3)^2}{(k_1+k_2)^2(k_1+k_3)^2(k_2+k_3)^2} \\
&\quad \cdot e^{(k_1+k_2+k_3)x-\left(k_1^3+k_2^3+k_3^3\right)t}.
\end{aligned} \tag{48}
$$

The singular three-soliton solutions follow immediately upon substituting (48) into (17).

4. The Third Coupled mKdV Equation

We consider now a third coupled mKdV equation given by

$$
\begin{aligned}
& u_t + \alpha(v^2 u_x - u^2 u_x) + \frac{\alpha}{4} u_{xxx} = 0, \\
& v_t + \alpha(v^2 v_x - u^2 v_x) + \frac{\alpha}{4} v_{xxx} = 0.
\end{aligned} \tag{49}
$$

where α is a constant. Our approach will run parallel to the approach used before, hence we skip details.

4.1. Multiple Soliton Solutions

The multi-soliton solutions of the coupled mKdV equation (49) are given by

$$
\begin{aligned}
u(x,t) &= R\left(\arctan\left(\frac{f(x,t)}{g(x,t)}\right)\right)_x = R\frac{f_x g - g_x f}{f^2+g^2}, \\
v(x,t) &= R_1\left(\arctan\left(\frac{f(x,t)}{g(x,t)}\right)\right)_x = R_1\frac{f_x g - g_x f}{f^2+g^2},
\end{aligned} \tag{50}
$$

where the auxiliary functions $f(x,t)$ and $g(x,t)$ are given earlier.

Proceeding as before and solving for R and R_1, we find

$$
\begin{aligned}
R &= \sqrt{\frac{6}{\gamma^2-1}}, \\
R_1 &= \gamma,
\end{aligned} \tag{51}
$$

where γ is a constant.

Combining (50)–(51) gives the single soliton solution

$$
\begin{aligned}
u(x,t) &= \frac{\sqrt{\frac{6}{\gamma^2-1}} k_1 e^{k_1\left(x-\frac{\alpha}{4}k_1^2 t\right)}}{1+e^{2k_1\left(x-\frac{\alpha}{4}k_1^2 t\right)}}, \\
v(x,t) &= \frac{\gamma\sqrt{\frac{6}{\gamma^2-1}} k_1 e^{k_1\left(x-\frac{\alpha}{4}k_1^2 t\right)}}{\alpha\beta\left(1+e^{2k_1\left(x-\frac{\alpha}{4}k_1^2 t\right)}\right)}.
\end{aligned} \tag{52}
$$

To determine the two-soliton solutions follow the discussion presented before to find that the phase shift a_{12} is given by

$$
a_{12} = \frac{(k_1-k_2)^2}{(k_1+k_2)^2}, \tag{53}
$$

and this can be generalized to

$$
a_{ij} = \frac{(k_i-k_j)^2}{(k_i+k_j)^2}, \quad 1 \le i < j \le 3. \tag{54}
$$

The two-soliton solutions are obtained by proceeding as before. It is interesting to point out that (53) does not show any resonant phenomenon [2] because the phase shift term a_{12} in (53) cannot be 0 or ∞ for $|k_1| \neq |k_2|$.

To determine the three-soliton solutions, we set

$$\begin{aligned} f(x,t) &= \mathrm{e}^{\theta_1} + \mathrm{e}^{\theta_2} + \mathrm{e}^{\theta_3} - a_{12}a_{13}a_{23}\mathrm{e}^{\theta_1+\theta_2+\theta_3} \\ &= \mathrm{e}^{k_1\left(x-\frac{\alpha}{4}k_1^2 t\right)} + \mathrm{e}^{k_2\left(x-\frac{\alpha}{4}k_2^2 t\right)} + \mathrm{e}^{k_3\left(x-\frac{\alpha}{4}k_3^2 t\right)} \\ &\quad - a_{12}a_{13}a_{23}\mathrm{e}^{(k_1+k_2+k3)x-\frac{\alpha}{4}\left(k_1^3+k_2^3+k_3^3\right)t}, \\ g(x,t) &= 1 - a_{12}\mathrm{e}^{\theta_1+\theta_2} - a_{13}\mathrm{e}^{\theta_1+\theta_3} - a_{23}\mathrm{e}^{\theta_2+\theta_3} \\ &= 1 - a_{12}\mathrm{e}^{(k_1+k_2)x-\frac{\alpha}{4}\left(k_1^3+k_2^3\right)t} \\ &\quad - a_{13}\mathrm{e}^{(k_1+k_3)x-\frac{\alpha}{4}\left(k_1^3+k_3^3\right)t} \\ &\quad - a_{23}\mathrm{e}^{(k_2+k_3)x-\frac{\alpha}{4}\left(k_2^3+k_3^3\right)t}, \end{aligned} \tag{55}$$

where the phase shifts a_{ij} are derived above in (54). The three-soliton solutions for the coupled mKdV equation (49) are obtained by substituting (55) into (50). It is clear that N-soliton solutions can be obtained for finite N, where $N \geq 1$.

4.2. Singular Soliton Solutions

The singular soliton solutions, single, two-soliton, and three-soliton solutions can be obtained in a like manner to the analysis presented above, hence we skip details.

5. Discussion

An analytic study was conducted on three coupled mKdV equations. The study confirmed the integrability of each coupled mKdV equation. Multiple-soliton solutions and multiple singular soliton solutions are formally derived. The analysis confirms the fact that certain equations which have N-soliton solutions, have simultaneously N-singular soliton solutions. The only change in the obtained results is the change in the coefficients of the dependent variable transformation. The phase shifts are the same for all coupled mKdV equations. The resonance phenomenon does not exist for any of these coupled mKdV equations.

[1] R. Hirota, Chaos Solitons Fractals **11**, 77 (2000).
[2] R. Hirota and M. Ito, J. Phys. Soc. Japan **52**(3), 744 (1983).
[3] M. Ito, J. Phys. Soc. Japan **49**(2), 771 (1980).
[4] R. Hirota and J. Satsuma, J. Phys. Soc. Japan **40**(2), 611 (1976).
[5] R. Hirota, Prog. Theor. Phys. **52**(5), 1498 (1974).
[6] R. Hirota, The Direct Method in Soliton Theory, Cambridge University Press, Cambridge 2004.
[7] R. Hirota, Phys. Rev. Lett. **27**(18), 1192 (1971).
[8] J. Hietarinta, J. Math. Phys. **28**(8), 1732 (1987).
[9] J. Hietarinta, J. Math. Phys. **28**(9), 2094 (1987).
[10] W. Hereman and A. Nuseir, Math. Comput. Simul. **43**, 13 (1997).
[11] W. Hereman and W. Zhuang, Symbolic computation of solitons with Macsyma, Computational and Applied Mathematics II: Differential Equations, 1992, 287 – 296.
[12] W. Hereman and W. Zhuang, A MACSYMA program for the Hirota method, Proceedings of the 13th IMACS World Congress on J. Computation and Applied Mathematics, 1991, 22 – 26.
[13] A. M. Wazwaz, Partial Differential Equations and Solitary Waves Theorem, Springer and HEP, Berlin 2009.
[14] A. M. Wazwaz, Can. J. Phys. **87**(12), 1227 (2010).
[15] A. M. Wazwaz, Appl. Math. Comput. **190**, 1198 (2007).
[16] A. M. Wazwaz, Commun. Nonlin. Sci. Numer. Simul. **12**(8), 1395 (2007).
[17] A. M. Wazwaz, Appl. Math. Comput. **192**, 479 (2007).
[18] A. M. Wazwaz, Appl. Math. Comput. **199**(1), 133 (2008).
[19] A. M. Wazwaz, Appl. Math. Comput. **200**, 437 (2008).
[20] A. M. Wazwaz, Appl. Math. Comput. **201**(1/2), 168 (2008).
[21] A. M. Wazwaz, Appl. Math. Comput. **201**(1/2), 489 (2008).
[22] A. M. Wazwaz, Appl. Math. Comput. **201**(1/2), 790 (2008).
[23] A. M. Wazwaz, Appl. Math. Comput. **204**(1), 20 (2008).
[24] A. M. Wazwaz, Appl. Math. Comput. **204**(1), 162 (2008).